Covariant Regularization of Nonlinear Spinorfield Quantum Theories and Probability Interpretation

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By a decomposition theorem a higher order nonlinear spinorfield equation can be transformed into a set of first order nonlinear spinorfield equations, i. e. into an auxiliary field formulation which allows canonical quantization. The quantum dynamics of the auxiliary fields is expressed in algebraic Schrödinger representation and admits only unphysical state spaces with indefinite metric. Regularization of the classical theory is transferred into quantum field theory by a noninvertible map from the corresponding auxiliary field state space into an associated physical state space, the metric of which is positive definite. For the effective dynamics in the physical state space probability current conservation is proved, and for physical states which describe composite particle configurations the existence of the state space is demonstrated.

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Introduction

First order nonlinear spinorfield equations in Minkowski space with canonical quantization and local selfinteraction are nonrenormalizable in perturbation theory. In former times this fact was considered as an absolute argument against the use of such equations. However, in the mean time the attitude towards such equations has changed. At present the strict refusal of nonrenormalizable theories has been given up, see for instance Weinberg [1], and spinorfield models are in accordance with the modern theoretical trends. In particular in quantum chromodynamics spinorfield models are used as effective theories in order to avoid the complications of the calculation with the full gluon-quark dynamics, cf. Vogl and Weise [2], and for such models mainly nonperturbative calculations are of interest. But independently of the special calculation scheme, by use of these models divergencies result if no regularization is applied. The most simple nonperturbative regularization is the cut-off regularization. This regularization has the drawback that it destroys the relativistic covariance of the theory, and as a consequence reduces the theory to the status of a low energy effective theory cf. [2]. So the question is: can one find a generally valid (perturbative and non-

perturbative) regularization scheme which avoids the violation of covariance and allows a physical interpretation without running into other difficulties?

In this paper we introduce a regularization scheme which replaces a first order nonlinear spinorfield equation by a higher order one. The fact that higher order wave equations lead to regularization in classical field theory was discovered by Bopp [3] and independently Podolski [4] in electrodynamics. The problem is to see what such a procedure means for a quantum field theory. For instance, in quantum electrodynamics Pauli-Villars regularization to all orders of perturbation theory can be formulated by higher order equations resulting from a corresponding Lagrangian similar to that of Bopp and Podolski, see Itzykson and Zuber [15]. But in contrast to quantum electrodynamics, first order spinorfields are nonrenormalizable and the regularization has to be applicable to nonperturbative calculations. This requires the development of a complete new regularization scheme and interpretation. It is the purpose of this paper to discuss this question.

As will be shown in the following, the spinorfield regularization method can be considered as a nonperturbative Pauli-Villars regularization. This designation indicates that in comparable cases this method

yields the same subtractions as the ordinary (perturbative) Pauli-Villars regularization, but that in contrast to the latter it needs not to be artificially added to the calculations but is a priori incorporated into the field theoretic formalism and can thus be applied in the nonperturbative range too. But before beginning with detailed deductions we explain the general lines of our method and review the specific difficulties which have to be mastered.

We base our discussion on a decomposition theorem of classical higher order nonlinear spinorfield equations demonstrated by Stumpf [5] for order $N = 3$ and by Grosser [6] for arbitrary N. This decomposition theorem enables one to transform such equations into a set of first order nonlinear spinorfield equations leading thus to the introduction of auxiliary fields which are similar to auxiliary fields introduced by perturbative Pauli-Villars regularization of renormalizable quantum field theories, cf. for instance Gupta [7]. But if these procedures are similar, this raises the question about the difference between these classes of theories.

The answer is that in the case of a renormalizable theory one can take the auxiliary field masses to infinity and obtain finite results that are insensitive to the limiting process, whereas this statement is false for a nonrenormalizable theory: In this case the limit to infinity is sensitive to the cut-off and can in general not be performed, see Collins [8]. Hence one has to expect that in nonlinear spinor theories with nonperturbative Pauli-Villars regularization the auxiliary field masses can be made very large but are not allowed to go to infinity. This fact has serious consequences. Irrespective of whether one considers renormalizable or nonrenormalizable theories, finite masses of the auxiliary fields bring in their train indefinite metric into the theory. If the limit to infinity can be performed, one can get rid of this indefinite metric, in the opposite case one has to deal with this question.

Thus for nonlinear spinorfields the prize of this kind of regularization is the introduction of indefinite metric into the theory. Apart from regularization, indefinite metric also appears in gauge theories for covariant gauges, where it is removed by constraints. But due to the absence of local gauge invariance in spinorfield models, this procedure cannot be imitated in this case. Nevertheless we shall show that by a suitable interpretation of regularization, a probability interpretation of such a theory is possible without scaling it down to the status of a low energy effective theory.

The mathematical elaboration runs as follows: The auxiliary fields are canonically quantized. Due to the indefiniteness of the auxiliary field metric we apply a GNS-like representation of the auxiliary field state space and integrate this representation by means of a map into generating functional states. Due to the Heisenberg dynamics of the auxiliary fields these generating functional states fulfill corresponding functional equations (algebraic Schrödinger representation). Nonperturbative regularization is defined by the introduction of a noninvertible map from the auxiliary field functionals to physical state functionals, whose set of matrixelements arises from the original set by summation over the auxiliary field indices. If this map is applied to the corresponding auxiliary field functional equation, in the limit of a pole-dipole configuration of the auxiliary fields, current conservation for the physical state functionals can be deduced which includes a probability interpretation of the physical functional state space.

All these steps are formal as long as it cannot be demonstrated that physical state functionals really exist, i.e. are finite. In order to show this we consider state spaces which describe composite particle (field) configurations. For such configurations it can be shown that the formally defined regularization really works. As an input into the calculation we use the free Feynman propagator of the auxiliary fields. In this case for the example of boson-bound states exact nonperturbative calculations can be performed, and an exact effective boson-boson dynamics can be derived. In this way, by direct construction of the (many) boson state space it is demonstrated that the effective boson theory allows a microscopic interpretation as a system of interacting bound states with a corresponding probability interpretation and an effective field dynamics derived from the original spinorfield.

As the construction of finite, i.e. regularized state spaces for composite particle (field) configurations was already treated in preceding papers, for brevity we refer to the literature and review only the results in the corresponding Section 7. Finally we want to emphasize that for closed systems the propagator should be selfconsistently calculated, and that the input of the free auxiliary field propagator into nonperturbative calculations is only meant as an example of a necessary propagator fixing which allows very transparent and nontrivial calculations. As a general reference

with respect to the composite particle dynamics and its physical interpretation we refer to Stumpf and Borne [9].

1. The Decomposition Theorem

The basic quantities of the model are two four-component Dirac spinor-isospinor fields $\Psi_{A\alpha}(x)$, where $\alpha = 1, \ldots, 4$ is the spinor index and $A = 1, 2$ the isospinor index. We postulate the following nonlinear field equation for these fields:

$$(i\hbar\gamma^\mu\partial_\mu - cm)^{\text{reg}}_{\alpha\beta}\delta_{AB}\Psi_{B\beta}(x) = g\big[\Psi_{A\alpha}(x) \tag{1}$$

$$\cdot\,\bar{\Psi}_{C\gamma}(x)\Psi_{C\gamma}(x) - \gamma^5_{\alpha\beta}\Psi_{A\beta}(x)\bar{\Psi}_{C\gamma}(x)\gamma^5_{\gamma\delta}\Psi_{C\delta}(x)\big],$$

which exhibits a scalar and a pseudoscalar coupling of the Dirac spinors and a scalar coupling of the isospin components. The superscript "reg" denotes the special kind of regularization already announced and will be defined below.

Equation (1) may be abbreviated as

$$(i\hbar\gamma^\mu\partial_\mu - cm)^{\text{reg}}_{\alpha\beta}\delta_{AB}\Psi_{B\beta}(x) \tag{2}$$

$$= gV_{\substack{ABCD\\ \alpha\beta\gamma\delta}}\Psi_{B\beta}(x)\bar{\Psi}_{C\gamma}(x)\Psi_{D\delta}(x),$$

where the vertex is defined by

$$V_{\substack{ABCD\\ \alpha\beta\gamma\delta}} := \sum_{h=1}^{2}\delta_{AB}\delta_{CD}v^h_{\alpha\beta}v^h_{\gamma\delta}, \tag{3}$$

$$v^1_{\alpha\beta} := \delta_{\alpha\beta}, \quad v^2_{\alpha\beta} := i\gamma^5_{\alpha\beta}.$$

The adjoint spinor-isospinor field

$$\bar{\Psi}_{A\alpha}(x) := \Psi^+_{B\beta}(x)\gamma^0_{\beta\alpha}\delta_{AB} \tag{4}$$

satisfies the adjoint equation

$$(-i\hbar\,{}^T\gamma^\mu\partial_\mu - cm)^{\text{reg}}_{\alpha\beta}\delta_{AB}\bar{\Psi}_{B\beta}(x) \tag{5}$$

$$= gV_{\substack{ABCD\\ \alpha\beta\gamma\delta}}\bar{\Psi}_{D\delta}(x)\Psi_{C\gamma}(x)\bar{\Psi}_{B\beta}(x).$$

The coupling constant g is real and will be fixed later. With respect to the choice of the vertex we refer to Grimm [10], who performed a systematic discussion of vertex expressions. As we will perform a quantization of the spinorfields as fermion fields, we have

to consider them on the "classical" level as anticommuting Grassmann variables; this leads to the antisymmetrization $V_{\substack{ABCD\\ \alpha\beta\gamma\delta}} = -V_{\substack{ADCB\\ \alpha\delta\gamma\beta}}$ of the vertex.

The invariance groups of the field equations (2) and (5) are the Poincaré group, a global $SU(2)$ isospin group and a global $U(1)$; the mass terms, however, break the chiral symmetries. The invariance of (2) and (5) with respect to the discrete C-, P- and T-transformations is established on the quantum level, i. e. if (2) and (5) are considered as equations for anticommuting operators. A detailed account of discrete transformations is given by Grimm [10].

Finally we explain the superscript "reg" of the Dirac operator in (2) and (5). It indicates that the two equations have to be regularized if applied in their quantized version. We define the regularization by

$$(i\hbar\gamma^\mu\partial_\mu - cm)^{\text{reg}}_{\alpha\beta} := \tag{6}$$

$$\big[(i\hbar\gamma^\mu\partial_\mu - cm_1)(i\hbar\gamma^\nu\partial_\nu - cm_2)(i\hbar\gamma^\rho\partial_\rho - cm_3)\big]_{\alpha\beta}$$

with

$$(m_i)_{\alpha\beta} := m_i\delta_{\alpha\beta}, \quad i = 1, 2, 3, \tag{7}$$

where m_i, $i = 1, 2, 3$, are real and $m_1 \neq m_2 \neq m_3$, i. e. we regularize (2) by a third order differential operator. For later applications we assume $m_1 > m_2 > m_3$. A theorem which decomposes a linear Dirac equation with higher order derivatives, i. e. (2) with $V \equiv 0$, into a set of linear Dirac equations with first order derivatives was first proved by Wildermuth [11]. The extension to the decomposition of nonlinear field equations is nontrivial.

The relation between a higher order differential (field) equation and first order equations with $V \neq 0$ can be established by the following "decomposition theorem" which was proved for $N = 3$ by Stumpf [5] and for general N by Grosser [6].

Proposition 1.1: Let $\{m_i, i = 1, \ldots, N, N \geq 2\}$ be a set of real parameters with $m_i \neq m_k$, $\forall\, i \neq k$ and D a complex variable, then the following statements hold:

i) Let $\Psi \equiv \Psi(x)$ be a solution of an Nth order equation

$$\prod_{i=1}^{N}(D - m_i)\Psi = V[\Psi], \tag{8}$$

where $V[\Psi]$ is a (nonlinear) interaction term, and let $\psi_i \equiv \psi_i(x)$, $i = 1, \ldots N$, be auxiliary fields which are defined by

$$\psi_i := \lambda_i \prod_{\substack{k=1 \\ k \neq i}}^{N} (D - m_k)\Psi, \quad i = 1, \ldots, N \qquad (9)$$

with

$$\lambda_i := \prod_{\substack{k=1 \\ k \neq i}}^{N} (m_i - m_k)^{-1}, \quad i = 1, \ldots, N. \qquad (10)$$

Then it holds

$$\Psi = \sum_{i=1}^{N} \psi_i, \qquad (11)$$

and the auxiliary fields are solutions of the first order equations

$$(D - m_i)\psi_i = \lambda_i V\left[\sum_{j=1}^{N} \psi_j\right], \quad i = 1, \ldots, N \qquad (12)$$

with no summation over i on the left-hand side.

ii) If the fields $\psi_i \equiv \psi_i(x)$, $i = 1, \ldots, N$, are solutions of (12) with λ_i of (10), and if $\Psi \equiv \Psi(x)$ is defined by (11), then the relations (9) hold and Ψ is a solution of (8).   □

The decomposition theorem is also valid for operator-valued D and can be easily extended to multicomponent equations. Thus we can apply it to the field equations (2) and (5) by defining the auxiliary fields

$$\psi_{A\alpha i}(x) := \lambda_i \prod_{\substack{k=1 \\ k \neq i}}^{3} (i\hbar\gamma^\mu\partial_\mu - cm_k)_{\alpha\beta}\delta_{AB}\Psi_{B\beta}(x), \qquad (13)$$

$$\bar{\psi}_{A\alpha i}(x) := \lambda_i \prod_{\substack{k=1 \\ k \neq i}}^{3} (-i\hbar\,{}^T\gamma^\mu\partial_\mu - cm_k)_{\alpha\beta}\delta_{AB}\bar{\Psi}_{B\beta}(x) \qquad (14)$$

with

$$\lambda_i := \prod_{\substack{k=1 \\ k \neq i}}^{3} (cm_i - cm_k)^{-1}. \qquad (15)$$

Then according to Proposition 1.1 we have the inverse relations

$$\Psi_{A\alpha}(x) = \sum_{i=1}^{3} \psi_{A\alpha i}(x), \quad \bar{\Psi}_{A\alpha}(x) = \sum_{i=1}^{3} \bar{\psi}_{A\alpha i}(x), \qquad (16)$$

and the auxiliary fields satisfy the first order differential equations

$$(i\hbar\gamma^\mu\partial_\mu - cm_i)_{\alpha\beta}\delta_{AB}\psi_{B\beta i}(x) \qquad (17)$$

$$= g\lambda_i V_{\substack{ABCD \\ \alpha\beta\gamma\delta}} \sum_{j,k,l=1}^{3} \psi_{B\beta j}(x)\bar{\psi}_{C\gamma k}(x)\psi_{D\delta l}(x)$$

and

$$(-i\hbar\,{}^T\gamma^\mu\partial_\mu - cm_i)_{\alpha\beta}\delta_{AB}\bar{\psi}_{B\beta i}(x) \qquad (18)$$

$$= g\lambda_i V_{\substack{ABCD \\ \alpha\beta\gamma\delta}} \sum_{j,k,l=1}^{3} \bar{\psi}_{D\delta l}(x)\psi_{C\gamma k}(x)\bar{\psi}_{B\beta j}(x)$$

with no summation over i on the left-hand sides. Due to (4) from (16) the relation

$$\bar{\psi}_{A\alpha i}(x) = \psi^+_{A\beta i}(x)\gamma^0_{\beta\alpha} \qquad (19)$$

follows, i.e. $\bar{\psi}$ has to be the adjoint auxiliary field spinor.

Finally we mention that the λ_i of (15) satisfy the Pauli-Villars regularization conditions, see Pauli and Villars [12], Rayski [13], Bogoliubov and Shirkov [14]:

$$\sum_{i=1}^{3} \lambda_i = 0, \quad \sum_{i=1}^{3} \lambda_i m_i = 0. \qquad (20)$$

Thus in combination with the Lagrange formulation for the auxiliary fields our regularization (6) turns out to be a nonperturbative Pauli-Villars regularization.

2. Canonical Spinorfield Quantization

In order to derive conservation laws and to carry out canonical quantization, we need the Lagrangian of the fields. This program can be successfully performed only if the field dynamics is based on first order differential equations. Thus the decomposition theorem serves as a means to apply spinorfield regularization and the canonical formalism simultaneously. So in the following all derivations and conclusions will be based on the auxiliary spinorfield dynamics given by (17) and (18). One can easily verify that (17) and

(18) follow by variation with respect to $\psi_{A\alpha i}(x)$ or $\bar{\psi}_{A\alpha i}(x)$ from an action with the Lagrangian density

$$\mathcal{L}[\psi, \bar{\psi}] = \mathcal{L}_{\text{kin}}[\psi, \bar{\psi}] + \mathcal{L}_{\text{int}}[\psi, \bar{\psi}] \tag{21}$$

$$= \sum_{i=1}^{3} \lambda_i^{-1} \bar{\psi}_{A\alpha i}(x)(i\hbar\gamma^\mu\partial_\mu - cm_i)_{\alpha\beta}\delta_{AB}\psi_{B\beta i}(x)$$

$$- \frac{g}{2}V_{\substack{ABCD \\ \alpha\beta\gamma\delta}} \sum_{i,j,k,l=1}^{3} \bar{\psi}_{A\alpha i}(x)\psi_{B\beta j}(x)\bar{\psi}_{C\gamma k}(x)\psi_{D\delta l}(x),$$

where $\mathcal{L}_{\text{kin}}$ has the structure of a Pauli-Villars regularized Lagrangian. But as far as $\mathcal{L}_{\text{int}}$ is concerned, in conventional quantum field theory $\mathcal{L}_{\text{int}}$ is regularized by hand and not automatically by the formalism itself, see, e. g., Itzykson and Zuber [15]. In contrast to this common usage, the Lagrangian (21) automatically provides also a regularization of the interaction term, and this regularization is independent of the kind of calculation. In particular it does not depend on the application of perturbation theory, as will be demonstrated in the following. Thus by the decomposition theorem we have achieved a nonperturbative Pauli-Villars regularization.

In order to examine the conserved quantities we define

$$\pi^\mu_{A\alpha i}(x) := \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi_{A\alpha i}(x)}. \tag{22}$$

Due to the translational invariance of the Lagrangian and its action, it follows that according to Noether's theorem the energy-momentum vector

$$P^\mu := \int \mathcal{T}^{0\mu}(r, t)\mathrm{d}^3 r \tag{23}$$

with the canonical energy-momentum tensor density

$$\mathcal{T}^{\mu\nu}(x) := \pi^\mu_{A\alpha i}(x)\partial^\nu \psi_{A\alpha i}(x) - \eta^{\mu\nu}\mathcal{L}(x) \tag{24}$$

is a conserved quantity.

The canonically conjugated momentum $\pi_{A\alpha i}(x) := \pi^0_{A\alpha i}(x)$ to $\psi_{A\alpha i}(x)$ can be calculated by means of (21) and (22) and reads

$$\pi_{A\alpha i}(x) = \frac{i\hbar}{\lambda_i}\bar{\psi}_{A\beta i}(x)\gamma^0_{\beta\alpha} = \frac{i\hbar}{\lambda_i}\psi^+_{A\alpha i}(x) \tag{25}$$

with no summation over i on the right-hand side. The hamiltonian density follows from $H(x) := c\mathcal{T}^{00}(x)$ and is given by

$$H := c\sum_{i=1}^{3} \lambda_i^{-1} \int \bar{\psi}_{A\alpha i}(r)\left(-i\hbar\gamma^k\partial_k + cm_i\right)_{\alpha\beta} \tag{26}$$

$$\cdot \delta_{AB}\psi_{B\beta i}(r)\mathrm{d}^3 r$$

$$+ c\frac{g}{2}V_{\substack{ABCD \\ \alpha\beta\gamma\delta}} \sum_{i,j,k,l=1}^{3} \int \bar{\psi}_{A\alpha i}(r)\psi_{B\beta j}(r)$$

$$\cdot \bar{\psi}_{C\gamma k}(r)\psi_{D\delta l}(r)\mathrm{d}^3 r$$

with $\psi_{A\alpha i}(r) \equiv \psi_{A\alpha i}(r, 0)$ With the decomposition theorem we are prepared to canonically quantize the regularized spinorfield which is dynamically characterized by the Lagrangian density (21). The canonical anticommutation relations read

$$\left[\pi_{A\alpha i}(r, t), \psi_{B\beta j}(r', t)\right]_+ = i\hbar\delta_{ij}\delta_{AB}\delta_{\alpha\beta}\delta(r - r'),$$

$$\left[\psi_{A\alpha i}(r, t), \psi_{B\beta j}(r', t)\right]_+ = 0, \tag{27}$$

$$\left[\pi_{A\alpha i}(r, t), \pi_{B\beta j}(r', t)\right]_+ = 0$$

for equal times. These equal time relations indicate that the complete CAR-algebra is already defined at an arbitrary, but fixed moment of time. The dynamical evolution of a quantum system is then described by a continuous group of automorphism of this algebra. If one substitutes (25) into (27) these relations go over into the spinorfield relations

$$\left[\psi^+_{A\alpha i}(r, t), \psi_{B\beta j}(r', t)\right]_+ = \lambda_i\delta_{ij}\delta_{AB}\delta_{\alpha\beta}\delta(r - r'),$$

$$\left[\psi_{A\alpha i}(r, t), \psi_{B\beta j}(r', t)\right]_+ = 0, \tag{28}$$

$$\left[\psi^+_{A\alpha i}(r, t), \psi^+_{B\beta j}(r', t)\right]_+ = 0.$$

Due to (20) not all λ_i can be positive quantities. Thus (28) shows that regularization enforces indefinite metric by producing alternating signs in the anticommutation relations of the fields. This feature of spinorfield theory will be discussed below.

The regularization effect can be drastically seen if one derives the anticommutation relations for the original field $\Psi_{A\alpha}(x)$ from (28). With (11) we obtain from (28) by means of (20)

$$\left[\Psi_{A\alpha}^+(\boldsymbol{r}, t), \Psi_{B\beta}(\boldsymbol{r}', t)\right]_+ \tag{29}$$

$$= \sum_{i,j=1}^{3} \left[\psi_{A\alpha i}^+(\boldsymbol{r}, t), \psi_{B\beta j}(\boldsymbol{r}', t)\right]_+$$

$$= \sum_{i,j=1}^{3} \lambda_i \delta_{ij} \delta_{AB} \delta(\boldsymbol{r} - \boldsymbol{r}') = 0,$$

$$\left[\Psi_{A\alpha}^+(\boldsymbol{r}, t), \Psi_{B\beta}^+(\boldsymbol{r}', t)\right]_+ \tag{30}$$

$$= \left[\Psi_{A\alpha}(\boldsymbol{r}, t), \Psi_{B\beta}(\boldsymbol{r}', t)\right]_+ = 0.$$

Thus the canonical quantization of the auxiliary fields is equivalent to a noncanonical quantization of the original field. The latter relations (29) were already assumed by Heisenberg [16], but without the background of a decomposition theorem and thus without relation to a strictly canonically quantized system. It is just the combination of our regularization with CAR-algebra and Hamilton formalism which allows a successful treatment of spinorfield dynamics.

In general a quantum theory is assumed to possess a representation in a Hilbert space, i.e. a vector space with a positive definite metric. As already mentioned in the introduction there are, however, prominent examples of theories which enlarge the concept of Hilbert space. Without referring to specific physical systems we first discuss the formal aspects of such generalizations. The properties of representations and corresponding representation spaces of CAR- or CCR-algebras, respectively, depend on the sign of the anticommutators or commutators defining the algebra. If all basic anticommutators or commutators have the same sign, then there exist representation spaces with positive definite metric. If these quantities have alternating signs, then the corresponding representation spaces are indefinite.

For instance, vector fields in covariant quantization possess basic commutators with alternating sign due to their dependence on the metrical tensor $\eta_{\mu\nu}$. Another example are the spinorial anticommutators (28), and, as announced in the introduction, we will just treat these kinds of systems. Hence the state spaces under consideration must be vector spaces $I\!H$ with indefinite metric, i.e. vector spaces with a scalarproduct $\langle \cdot, \cdot \rangle : I\!H \times I\!H \to \mathbb{C}$, and with the properties

i) $\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$
   for all $\alpha_1, \alpha_2 \in \mathbb{C}$, $x_1, x_2, y \in I\!H$,

ii) $\langle y, x \rangle = \langle x, y \rangle^*$ for all $x, y \in I\!H$,

iii) $\langle x, x, \rangle \leq 0$ for at least one $x \in I\!H$.

Furthermore to be able to make nontrivial statements about the state space, in particular to introduce a basis in $I\!H$, we assume $I\!H$ to be a Krein space, i. e., $I\!H$ is nondegenerate and has the decomposition

$$I\!H = I\!H_+ \oplus I\!H_-$$

with $I\!H_+ \subset P^{++}$, $I\!H_- \subset P^{--}$, where

$$P^{++} := \left\{ x \in I\!H, \langle x, x \rangle > 0 \text{ or } x = 0 \right\},$$

$$P^{--} := \left\{ x \in I\!H, \langle x, x \rangle < 0 \text{ or } x = 0 \right\},$$

and where $I\!H_+$ and $I\!H_-$ are assumed to be intrinsic complete, while $\oplus$ denotes the orthogonal direct sum. The decomposition is referred to as fundamental decomposition. We assume that the Krein spaces are spanned by a set of generalized eigenstates for a set of commuting selfadjoint observables. With respect to the latter the following statements can be verified for Krein spaces: The adjoint of an operator with dense domain of definition in $I\!H$ exists and is unique; the adjunction is an involution on the representation of the algebra, cf. Bognar [17]. Furthermore in such spaces suitable representations of the physical transformation groups exist, see, e. g., Bracci, Morchio and Strocchi [18] or Schlieder [19].

In general the eigenstates of a set of commuting observables on an indefinite state space need not constitute a basis of this space. However, in Krein space one can find generalized eigenstates enlarging the set of eigenstates to a set of basis states of $I\!H$, cf. Bognar [17]. Hence all the physics and "non-physics" can be covered by this construction and one can base the probability interpretation on it. Another complication in indefinite state spaces is the fact that the eigenvalues of selfadjoint operators, in our case the momentum and spin parameters p and s may take complex values. However, it is easy to show that in Krein space the eigenvalues of selfadjoint operators for eigenstates with a nonvanishing norm are real. Due to the non-degeneracy of the Krein space in particular the eigenvalues of the states spanning the physical subspace $I\!H_{ph}$ are real. Hence quantum theory in $I\!H_{ph}$ can be developed as usual.

Of course for a suitable physical interpretation of the theory the physical subspace must be positiv definite. In the wellknown examples of gauge theories the

subspace $\mathrm{I\!H}_+$ is directly defined to be the physical subspace, i.e. $\mathrm{I\!H}_{ph} \equiv \mathrm{I\!H}_+$ and by suitable subsidiary conditions the physical processes are confined not to leave this subspace. In the case of the regularized spinorfield such subsidiary conditions, which are connected with gauge invariance, do not exist. *Hence we have to apply a new method in order to construct the physical subspace.* This method consists in a single valued mapping from $\mathrm{I\!H}$ to $\mathrm{I\!H}_{phys}$ and it is $\mathrm{I\!H}_{phys} \neq \mathrm{I\!H}_+$. Naturally by this mapping information about the full statespace $\mathrm{I\!H}$ is lost. But $\mathrm{I\!H}$ is unphysical and it cannot be the task of a physically interpretable theory to deliver information about unphysical state spaces. We will discuss the construction of $\mathrm{I\!H}_{phys}$ in Sects. 5, 6, and 7.

3. Superindexing

Formally, superindexing is a very compact notation for the spinorfields and their equations. Later on, however, it will turn out that superindexing is indispensable for an appropriate phenomenological interpretation, i.e. for the physical understanding of the theory.

We introduce the charge conjugated spinorfields

$$\psi^c_{A\alpha i}(x) := C_{\alpha\beta}\bar{\psi}_{A\beta i}(x) \tag{31}$$

with the charge conjugation matrix C, and rewrite (17) and (18) in terms of ψ and ψ^c. This gives

$$(i\hbar\gamma^\mu\partial_\mu - cm_i)_{\alpha\beta}\delta_{AB}\psi_{B\beta i}(x) \tag{32}$$

$$= g\lambda_i V_{\underset{\alpha\beta\gamma\delta}{ABCD}} C^{-1}_{\gamma\epsilon} \sum_{j,k,l=1}^{3} \psi_{B\beta j}(x)\psi^c_{C\epsilon k}(x)\psi_{D\delta l}(x)$$

and

$$(i\hbar\gamma^\mu\partial_\mu - cm_i)_{\alpha\beta}\delta_{AB}\psi^c_{B\beta i}(x) = -g\lambda_i V_{\underset{\alpha'\beta'\gamma\delta'}{ABCD}} \tag{33}$$

$$\cdot C_{\alpha\alpha'}C^{-1}_{\beta'\beta}C^{-1}_{\delta'\delta} \sum_{j,k,l=1}^{3} \psi^c_{B\beta j}(x)\psi_{C\gamma k}(x)\psi^c_{D\delta l}(x)$$

with no summation over i on the left-hand sides. From (29) it follows that also $\Psi^c \equiv \sum_i \psi^c_i$ and Ψ anticommute for equal times. Hence the vertices in (32) and (33) have the same antisymmetry properties as the old ones.

Equations (32) and (33) can now be combined into one superspinor equation if we introduce superspinors by the definition

$$\psi_{A\alpha i\Lambda}(x) := \begin{cases} \psi_{A\alpha i}(x), & \Lambda = 1 \\ \psi^c_{A\alpha i}(x), & \Lambda = 2. \end{cases} \tag{34}$$

Together with the superindex $Z := (A, \alpha, i, \Lambda)$, where

$$\begin{aligned}
A &= 1, 2 & &\text{(isospinor index)} \\
\alpha &= 1, 2, 3, 4 & &\text{(spinor index)} \\
i &= 1, 2, 3 & &\text{(auxiliary field index)} \\
\Lambda &= 1, 2 & &\text{(superspinor index)},
\end{aligned}$$

(32) and (33) can be rewritten as

$$\left(D^\mu_{Z_1 Z_2}\partial_\mu - m_{Z_1 Z_2}\right)\psi_{Z_2}(x) \tag{35}$$

$$= \hat{U}_{Z_1 Z_2 Z_3 Z_4}\psi_{Z_2}(x)\psi_{Z_3}(x)\psi_{Z_4}(x)$$

with

$$D^\mu_{Z_1 Z_2} := i\hbar\gamma^\mu_{\alpha_1\alpha_2}\delta_{A_1 A_2}\delta_{i_1 i_2}\delta_{\Lambda_1 \Lambda_2}, \tag{36}$$

$$m_{Z_1 Z_2} := cm_{i_1}\delta_{A_1 A_2}\delta_{\alpha_1\alpha_2}\delta_{i_1 i_2}\delta_{\Lambda_1 \Lambda_2},$$

$$\hat{U}_{Z_1 Z_2 Z_3 Z_4} := \sum_{h=1}^{2} g\lambda_{i_1} B_{i_2 i_3 i_4} v^h_{\alpha_1\alpha_2}\delta_{A_1 A_2}\delta_{\Lambda_1 \Lambda_2}$$

$$\cdot (v^h C)_{\alpha_3\alpha_4}\delta_{A_3 A_4}\delta_{\Lambda_3 1}\delta_{\Lambda_4 2},$$

$$B_{i_2 i_3 i_4} = 1 \quad \text{for } i_2, i_3, i_4 = 1, 2, 3,$$

where $B_{i_2 i_3 i_4}$ is a formal quantity expressing the auxiliary field summation in the vertex, and with

$$\hat{U}_{Z_1 Z_2 Z_3 Z_4} \equiv \hat{U}_{Z_1 [Z_2 Z_3 Z_4]} \tag{37}$$

due to the total anticommutativity of the sum of all auxiliary fields in the vertex.

In contrast to the sum of all auxiliary fields, the single auxiliary fields fulfill nontrivial anticommutation relations (28) which read in superindex notation

$$\left[\psi_Z(\boldsymbol{r}, t), \psi_{Z'}(\boldsymbol{r}', t)\right]_+ = A_{ZZ'}\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{38}$$

with

$$A_{ZZ'} := \lambda_i\delta_{ii'}\delta_{AA'}\sigma^1_{\Lambda\Lambda'}(C\gamma^0)_{\alpha\alpha'}, \tag{39}$$

no summation over i on the left-hand side and $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Moreover as often as possible we still use a more compact notation by combining (Z, x) into only one general index $I := (Z, x)$ and denote the underlying index set with $\mathcal{I}$. In this case (38) can be rewritten as

$$[\psi_I, \psi_{I'}]_+^{t=t'} = A_{II'} \tag{40}$$

with

$$A_{II'} := A_{ZZ'}\delta(\mathbf{r} - \mathbf{r}'). \tag{41}$$

In a similar way equation (35) can be compactified and reads

$$K_{I_1 I_2}\psi_{I_2} = W_{I_1 I_2 I_3 I_4}\psi_{I_2}\psi_{I_3}\psi_{I_4}, \tag{42}$$

where the following abbreviations are used:

$$K_{I_1 I_2} := \left(D^\mu\partial_\mu - m\right)_{I_1 I_2} \tag{43}$$

with

$$D^\mu_{I_1 I_2} := D^\mu_{Z_1 Z_2}\delta(x_1 - x_2), \tag{44}$$

$$m_{I_1 I_2} := m_{Z_1 Z_2}\delta(x_1 - x_2), \tag{45}$$

and

$$W_{I_1 I_2 I_3 I_4} := \hat{U}_{Z_1 Z_2 Z_3 Z_4}\delta(x_1 - x_2) \tag{46}$$
$$\cdot \; \delta(x_1 - x_3)\delta(x_1 - x_4),$$

and where the summation convention is assumed to range over the whole set of indices in $I = (Z, x)$.

In the course of our investigations we will also need a version of (42) where (42) is resolved with respect to ψ_I, and t is considered as an evolution parameter. In this case we transform (42) into

$$i\frac{\hbar^2}{c}\frac{\partial}{\partial t}\psi_{I_1} = \hat{K}_{I_1 I_2}\psi_{I_2} + \hat{W}_{I_1 I_2 I_3 I_4}\psi_{I_2}\psi_{I_3}\psi_{I_4}, \tag{47}$$

where all field operators have to be taken at the same time t, and exclude t from the summation convention. The corresponding definitions are

$$\hat{K}_{I_1 I_2} := iD^0_{Z_1 Z_3}\left(D^k_{Z_3 Z_2}\partial_k - m_{Z_3 Z_2}\right)\delta(\mathbf{r}_1 - \mathbf{r}_2), \tag{48}$$

$$\hat{W}_{I_1 I_2 I_3 I_4} := -iD^0_{Z_1 Z_5}\hat{U}_{Z_5 Z_2 Z_3 Z_4}\delta(\mathbf{r}_1 - \mathbf{r}_2), \tag{49}$$
$$\cdot \; \delta(\mathbf{r}_1 - \mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r}_4)$$

and the summation convention is assumed to range over the indices $(Z, \mathbf{r})$ in $I = (Z, \mathbf{r}, t)$.

4. Algebraic Schrödinger Representation

Probability conservation for ordinary Dirac equations is proved by using the Hamilton form of the Dirac equation. The analogous formulation in quantum field theory is the algebraic Schrödinger representation. The explicit evaluation of this representation depends on the ability to define appropriate basis vector systems. We emphasize that in quantum field theory this must be done in accordance with general ideas of algebraic representation theory and axiomatic quantum field theory. The corresponding construction of basis vector systems in Algebraic Quantum Theory is the GNS-construction (Gelfand-Naimark-Segal construction, cf. Bratelli and Robinson [20]), a generalization of the Fock space construction.

The GNS-construction provides genuine representations of the basic field operator algebra. As already mentioned, this algebra is already complete (or even overcomplete) at any spacelike hyperplane. In our case this is expressed by the equal time anticommutation relations (28). For simplicity we will only consider spacelike hyperplanes defined by $t = $ const.

The GNS-construction is based on the existence of cyclic vectors. Excluding the groundstates of thermofield dynamics we assume that a suitable cyclic vector is given by the physical groundstate $|0\rangle \in \mathbb{H}$. Then the set of basis vectors is generated by the application of monomials of field operators to the groundstate $|0\rangle$, and according to the completeness, or even overcompleteness of the algebra on any spacelike hyperplane we confine these products to equal times in their coordinates.

In the following we use the abbreviation

$$\left(\psi_{I_1}\ldots\psi_{I_n}\right)_t := \psi_{Z_1}(\mathbf{r}_1, t)\ldots\psi_{Z_n}(\mathbf{r}_n, t) \tag{50}$$

for field operator products at equal times which are the generating elements of our GNS-construction. For spinorfields the symmetric products can be excluded and we define for antisymmetric products the corresponding GNS-basis vectors by the set

$$\mathbb{B} := \left\{ \mathcal{A}\left(\psi_{I_1}\ldots\psi_{I_n}\right)_t |0\rangle, \; n \in \mathbb{N} \right\}, \tag{51}$$

where $\mathcal{A}$ means antisymmetrization in $I_1\ldots I_n$, i.e.

$$\mathcal{A}\left(\psi_{I_1}\ldots\psi_{I_n}\right)_t := \sum_{p\in S_n}\frac{1}{n!}\mathrm{sgn}(p)\psi_{I_{p_1}}\ldots\psi_{I_{p_n}}. \tag{52}$$

With respect to the set $\mathbb{B}$ any state $|a\rangle \in \mathbb{H}$ can be represented by

$$|a\rangle = \sum_n \sum_{I_1 \dots I_n} \frac{1}{n!} \rho_t^{(n)}(I_1 \dots I_n|a) \mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t |0\rangle,$$
(53)

where $\rho_t^{(n)}(I_1 \dots I_n|a)$ is antisymmetric in $I_1 \dots I_n$ for $t_1 = \dots = t_n = t$.

However in algebraic Schrödinger representation the states (53) are not directly calculated. Rather one considers the set of matrixelements

$$\tau_t^{(n)}(a) := \langle 0|\mathcal{A}\left(\psi_{I_1} \dots \psi_{I_n}\right)_t |a\rangle, \quad n = 1 \dots \infty. \tag{54}$$

In Fock space such objects like (54) are referred to as "wavefunctions". In GNS-spaces we can maintain this designation, but have to pay attention to the fact that the state norms are not as simple as in Fock space.

In order to obtain a compact formulation of the field dynamics we define for the sets of matrixelements (54) generating functionals. To this end we introduce the generators of a CAR–algebra on the hyperplane $t = \text{const}$ by

$$j_I^t := j_Z^t(r) \quad \text{and} \quad \partial_I^t := \partial_Z^t(r), \tag{55}$$

and mark them with the superscript t. *In this case we understand the symbol I to be an abbreviation for $I = (Z, r, t)$ with the fixed parameter t.* In particular the summation convention does not include the parameter t in the following.

The anticommutators of the CAR-generators on the hyperplane are then given by

$$\left[j_I^t, j_{I'}^t\right]_+ = \left[\partial_I^t, \partial_{I'}^t\right]_+ = 0, \tag{56}$$

$$\left[j_I^t, \partial_{I'}^t\right]_+ = \delta_{ZZ'}\delta(r - r'), \tag{57}$$

and their Fock space representation results from $\left(j_I^t\right)^+ |0\rangle_F = \partial_I^t|0\rangle_F = 0$. The state $|0\rangle_F$ is again assumed to be a cyclic state of a corresponding functional space $\mathbb{K}_F^t$. For brevity we do not explicitly introduce the transformation properties of the generators (55) but refer to Stumpf and Borne [9].

We define functional states on the hyperplane by

$$|\mathcal{A}(j^t; a)\rangle := \sum_{n=1}^{\infty} \frac{i^n}{n!} \sum_{I_1 \dots I_n} \tau_t^{(n)}(I_1 \dots I_n|a) \tag{58}$$

$$\cdot j_{I_1}^t \dots j_{I_n}^t |0\rangle_F$$

with

$$\tau_t^{(n)}(I_1 \dots I_n|a) := \langle 0|\mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t |a\rangle, \tag{59}$$

and in this way we associate to any state $|a\rangle \in \mathbb{H}$ a functional state $|\mathcal{A}(j^t; a)\rangle \in \mathbb{K}_F^t$.

We observe that the Heisenberg formula $i\hbar\dot{\mathbf{O}} = [\mathbf{O}, \mathbf{H}]_-$ can be applied to any observable $\mathbf{O}$, which is given as a power series of field operators, in particular to the antisymmetrized products $\mathbf{A}_n$. This leads to

$$i\hbar\frac{\partial}{\partial t}\mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t = \left[\mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t, \mathbf{H}\right]_-, \tag{60}$$

or in combination with the eigenstates of $\mathbf{H}$ to

$$i\hbar\frac{\partial}{\partial t}\langle 0|\mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t |a\rangle \tag{61}$$

$$= (E_a - E_0)\langle 0|\mathcal{A}(\psi_{I_1} \dots \psi_{I_n})_t |a\rangle.$$

The evaluation of these relations can be performed by bringing the field dynamics into play. In Stumpf, Fauser, and Pfister [21] and in Stumpf and Borne [9] this evaluation was done with the replacement of the time derivatives in (61) by the corresponding equations of motion for the ψ-field and subsequent antisymmetrization. But the most simple way is the direct evaluation of the right–hand side of (60) in functional space as was demonstrated by Stumpf and Pfister [22]. This leads to the following proposition:

Proposition 4.1: The set of equations (60) for $n = 1, 2, \dots, \infty$ can be equivalently expressed by the functional equation

$$E_{p0}|\mathcal{A}(j^t; a)\rangle = \left[\hat{K}_{I_1 I_2} j_{I_1}^t \partial_{I_2}^t - \hat{W}_{I_1 I_2 I_3 I_4} j_{I_1}^t \right. \tag{62}$$

$$\left. \cdot \left(\partial_{I_4}^t \partial_{I_3}^t \partial_{I_2}^t + \frac{1}{4} A_{I_4 J_1} A_{I_3 J_2} j_{J_1}^t j_{J_2}^t \partial_{I_2}^t\right)\right]|\mathcal{A}(j^t; a)\rangle. \quad \square$$

This proposition rests on the evaluation of the formula

$$E_{p0}|\mathcal{A}(j^t, a)\rangle = \left[H\left(\frac{1}{i}\partial^t + \frac{i}{2}Aj^t\right) \tag{63}\right.$$

$$\left. - H\left(\frac{1}{i}\partial^t - \frac{i}{2}Aj^t\right)\right]|\mathcal{A}(j^t, a)\rangle.$$

The functional equation (62) holds for any representation. It contains no elements which are referred to a special representation. Such representation fixing

elements can be introduced by general normalorderings. In the simplest case propagators are used for normalordering. Thus for any state $|a\rangle \in \mathbb{IH}$ we define normalordered functional states on a spacelike hyperplane $t = \text{const}$ by

$$|\mathcal{F}(j^t; a)\rangle := \exp\left[\frac{1}{2}\int j^t_{Z_1}(r_1)F^t_{Z_1 Z_2}(r_1, r_2)\right. \quad (64)$$

$$\left. \cdot j^t_{Z_2}(r_2)\mathrm{d}^3 r_1 \mathrm{d}^3 r_2\right]|\mathcal{A}(j^t; a)\rangle$$

$$=: \sum_{n=1}^{\infty}\frac{i^n}{n!}\varphi^{(n)}_t(I_1\dots I_n|a)j^t_{I_1}\dots j^t_{I_n}|0\rangle_{\mathrm{F}}$$

with the propagator

$$F^t_{Z_1 Z_2}(r_1, r_2) := \langle 0|\mathcal{A}[\psi^f_{Z_1}(r_1, t)\psi^f_{Z_2}(r_2, t)]|0\rangle, \quad (65)$$

where $\psi^f_Z(r, t)$ is a solution of equation (35) for $\hat{U}_{Z_1 Z_2 Z_3 Z_4} \equiv 0$.

Proposition 4.2: The normaltransform of (62) reads

$$E_{p0}|\mathcal{F}(j^t; a)\rangle = \frac{ic}{\hbar}\int \mathrm{d}^3 r j^t_{Z_0}(r)D^0_{Z_0 Z_1}$$

$$\cdot \left\{(D^k\partial_k - m)_{Z_1 Z_2}\partial^t_{Z_2}(r) + \hat{U}_{Z_1 Z_2 Z_3 Z_4}\right. \quad (66)$$

$$\cdot \left[d^t_{Z_4}(r)d^t_{Z_3}(r)d^t_{Z_2}(r)\right.$$

$$\left.\left. + \frac{1}{4}A_{Z_4 Z_5}j^t_{Z_5}(r)A_{Z_3 Z_6}j^t_{Z_6}(r)d^t_{Z_2}(r)\right]\right\}|\mathcal{F}(j^t; a)\rangle$$

with

$$d^t_{Z_1}(r_1) := \partial^t_{Z_1}(r_1) - \int F^t_{Z_1 Z_2}(r_1, r_2)j^t_{Z_2}(r_2)\mathrm{d}^3 r_2. (67)$$

$\square$

Then (66) can be equivalently written in the form

$$E_{p0}|\mathcal{F}(j^t; a)\rangle = \mathcal{H}_{\mathrm{F}}(j^t, \partial^t)|\mathcal{F}(j^t; a)\rangle \quad (68)$$

with the functional Hamilton operator

$$\mathcal{H}_{\mathrm{F}}(j^t, \partial^t) := j^t_{I_1}\hat{K}_{I_1 I_2}\partial^t_{I_2} + \hat{W}_{I_1 I_2 I_3 I_4} \quad (69)$$

$$\cdot \left[j^t_{I_1}\partial^t_{I_4}\partial^t_{I_3}\partial^t_{I_2} - 3F^t_{I_4 K_1}j^t_{I_1}j^t_{K_1}\partial^t_{I_3}\partial^t_{I_2} + (3F^t_{I_4 K_1}F^t_{I_3 K_2}\right.$$

$$+ \frac{1}{4}A_{I_4 K_1}A_{I_3 K_2})j^t_{I_1}j^t_{K_1}j^t_{K_2}\partial^t_{I_2} - (F^t_{I_4 K_1}F^t_{I_3 K_2}$$

$$\left. + \frac{1}{4}A_{I_4 K_1}A_{I_3 K_2})F^t_{I_2 K_3}j^t_{I_1}j^t_{K_1}j^t_{K_2}j^t_{K_3}\right].$$

5. Regularization and Field Dynamics

So far we have formally developed the Algebraic Schrödinger representation, but we have nothing said about regularization. Due to the necessity to perform a well-defined canonical quantization of the classical regularized spinorfield model (1), its quantum field theory was formulated by way of auxiliary fields and elementary calculations show that the corresponding GNS-matrixelement are partially singular or divergent and that the state space metric becomes indefinite, see Stumpf and Borne [9].

Obviously these difficulties can only be removed if the regularization of the original "classical" spinorfield equation can be transferred to its quantum field theory formulation. Considering perturbative Pauli-Villars regularization we remember that in the process of calculating vacuum expectation values the singular propagators are replaced by additive regularized expressions. Can one find a nonperturbative counterpart in the Algebraic Schrödinger Representation? In the absence of a selfconsistency calculation we assumed the free auxiliary field propagator to be a reasonable approximation, see (64) and (65). We write it in the form

$$F_{\substack{\alpha_1\alpha_2 \\ \kappa_1\kappa_2 \\ i_1 i_2}}(x_1, x_2) = -i\lambda_{i_1}\delta_{i_1 i_2}(\gamma^5)_{\kappa_1\kappa_2}\frac{1}{(2\pi)^4} \quad (70)$$

$$\cdot \int \mathrm{d}^4 p\, e^{ip(x_1 - x_2)}\left(\frac{1}{\gamma^\mu p_\mu + m_{i_1}}C\right)_{\alpha_1\alpha_2}$$

with $\kappa = (A, \Lambda)$. Summation over i_1, i_2 and application of the decomposition theorem yields

$$\hat{F}_{\substack{\alpha_1\alpha_2 \\ \kappa_1\kappa_2}}(x_1, x_2) = \sum_{i_1, i_2}F_{\substack{\alpha_1\alpha_2 \\ \kappa_1\kappa_2 \\ i_1 i_2}}(x_1, x_2) \quad (71)$$

$$= -i(\gamma^5)_{\kappa_1\kappa_2}\frac{1}{(2\pi)^4}\int \mathrm{d}^4 p\, e^{ip(x_1 - x_2)}$$

$$\cdot \left(\prod_{i=1}^{3}\frac{1}{\gamma^\mu p_\mu + m_{i_1}}C\right)_{\alpha_1\alpha_2},$$

i. e. a regularized fermion propagator.

From this result it follows imperatively that the only way to achieve a nonperturbative regularization consists in summing over the auxiliary field indices. Therefore we define the physical, i. e. regularized normalordered state amplitudes $\hat{\varphi}^{(n)}$ by

$$\hat{\varphi}^{(n)}_{\substack{\alpha_1\ldots\alpha_n\\ \kappa_1\ldots\kappa_n}} (\boldsymbol{r}_1 \ldots \boldsymbol{r}_n | a) := \sum_{\substack{i_1\ldots i_n}} \varphi^{(n)}_{\substack{\alpha_1\ldots\alpha_n\\ \kappa_1\ldots\kappa_n\\ i_1\ldots i_n}} (\boldsymbol{r}_1 \ldots \boldsymbol{r}_n | a),$$

(72)

in accordance with the regularized propagator (71). One immediately realizes that $\hat{\varphi}^{(n)}$ has the same transformation properties as the original $\varphi^{(n)} \ \forall \ n$.

Can one derive an associated dynamical law for (72), i.e. the one-time physical amplitudes? To answer this question, we consider the projection of (68) into configuration space. This gives the following set of equation for the normalordered G.N.S.-matrix elements:

$$\Delta E \varphi_n(I_1 \ldots I_n) = \sum_{l=1}^{n} \hat{K}_{I_l J} \varphi_n(I_1 \ldots I_{l-1} J I_{l+1} \ldots I_n)$$

$$+ \frac{1}{n!} \sum_{l=1}^{n} \sum_{\lambda_1\ldots\lambda_n} (-1)^P \hat{W}_{I_{\lambda_l} J_2 J_3 J_4}$$

$$\cdot \varphi_{n+2}(J_3 J_4 I_{\lambda_1} \ldots I_{\lambda_{l-1}} J_2 I_{\lambda_{l+1}} \ldots I_{\lambda_n})$$

$$+ \frac{1}{n!} \sum_{\lambda_1\ldots\lambda_n} (-1)^P \hat{W}_{I_{\lambda_l} J_2 J_3 J_4} \qquad (73)$$

$$\cdot \Big\{ \sum_{l=2}^{n} (-1)^l F_{J_4 I_{\lambda_{l-1}}} \varphi_n(J_2 J_3 I_{\lambda_2} \ldots I_{\lambda_{l-2}} I_{\lambda_l} \ldots I_{\lambda_n})$$

$$+ N_{J_4 J_3 I_{\lambda_2} I_{\lambda_3}} (n-2)(n-1)n \varphi_{n-2}(J_2 I_{\lambda_4} \ldots I_{\lambda_n})$$

$$+ N_{J_4 J_3 J_2 I_{\lambda_2} I_{\lambda_3} I_{\lambda_4}} (n-3)(n-2)(n-1)n$$

$$\cdot \varphi_{n-4}(I_{\lambda_5} \ldots I_{\lambda_n}) \Big\}$$

with

$$N_{J_3 J_4 I_2 I_3} := 3 F_{J_4 I_2} F_{J_3 I_3} + \frac{1}{4} A_{J_4 I_2} A_{J_3 I_3} \qquad (74)$$

and

$$N_{J_4 J_3 J_2 I_2 I_3 I_4} := N_{J_3 J_4 I_2 I_3} F_{J_2 I_4} \qquad (75)$$

We now decompose the Index $I := Z, \boldsymbol{r}$ with $Z := (A, \alpha, \Lambda, i)$ into $Z = (z, i)$ and $z = (A, \alpha, \Lambda)$ and the index $J = U, \boldsymbol{r}$ into $U = (p, j)$ etc.. In accordance with (48) and (49) we then have $D^\mu_{Z_1 Z_2} = D^\mu_{z_1 z_2} \delta_{i_1 i_2}$ and

$$\hat{W}_{I_\lambda J_2 J_3 J_4} = \sum_h D^0_{z_\lambda z} \lambda_{i_\lambda} B_{j_2 j_3 j_4} V^h \begin{pmatrix} \boldsymbol{r}_\lambda \boldsymbol{r}_1 \boldsymbol{r}_2 \boldsymbol{r}_3 \\ z p_1 p_2 p_3 \end{pmatrix}.$$

(76)

In order to study the time evolution of the regularized physical functions (72) we summarize the set of equations (73) over $i_1 \ldots i_n$. Due to (20) and (76) all terms containing $\hat{W}$ vanish and one obtains

$$\Delta E \hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \ldots \boldsymbol{r}_n \\ z_1 \ldots z_n \end{pmatrix} \qquad (77)$$

$$= \sum_{l=1}^{n} i D^0_{z_l z'} D^k_{z' z} \partial_k(\boldsymbol{r}_l) \hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \ldots \boldsymbol{r}_l \ldots \boldsymbol{r}_n \\ z_1 \ldots z \ldots z_n \end{pmatrix}$$

$$- \sum_{l=1}^{n} \sum_{i_1\ldots i_n} i D^0_{z_l z} m_{i_l} \varphi_n \begin{pmatrix} \boldsymbol{r}_1 \ldots \boldsymbol{r}_l \ldots \boldsymbol{r}_n \\ z_1 \ldots z \ldots z_n \\ i_1 \ldots i_l \ldots i_n \end{pmatrix}.$$

Evidently (77) is no selfconsistent equation for the calculation of $\hat{\varphi}_n$. Rather (77) brings about a connection between regularized physical amplitudes and auxiliary amplitudes. This result can be explained in the following way: If one applies the regularization prescription (72) to the antisymmetrized field monomials (59), due to (16) these monomials are transformed into those for the original Ψ-fields and due to their vanishing anticommutation relations (29) and (30) no Heisenberg equation of motion (60) can exist, i.e. no algebraic Schrödinger representation can be calculated. This, of course, holds for their normaltransforms (72) too. Therefore (77) has to be considered as an exact subsidiary condition only, analogously to the Lorentz condition in quantum electrodynamics.

Hence the auxiliary canonical field formulation in Algebraic Schrödinger representation is indispensible for the calculation of the dynamical evolution of the regularized spinorfield.

But how can one work with the latter representation if it produces a singular function? This problem will be treated in Sect. 7, and irrespective of this problem we can draw a first conclusion from (77). In accordance with the assumption of a Krein space for the representation of the auxiliary field space the spectrum of eigensolutions of (73) must be real, i.e. $\Delta E_k \equiv \epsilon_k, \ k = 1 \ldots \infty$ are real values. In addition, for the further treatment of (77) we assume that the set of eigenvalues $\{\epsilon_k\}$ does not depend on the auxiliary field indices i. This assumption will be justified in Sect. 7 too.

We now rewrite (77) in a more suitable form. We define the spintensors

$$\alpha^k(l) := \delta_{\alpha_1 \alpha'_1} \ldots \alpha^k_{\alpha_l \alpha'_l} \ldots \delta_{\alpha_n \alpha'_n},$$
$$\beta(l) := \delta_{\alpha_1 \alpha'_1} \ldots \beta_{\alpha_l \alpha'_l} \ldots \delta_{\alpha_n \alpha'_n} \tag{78}$$

for all integers $n = 1 \ldots \infty$. For abbreviation we suppress all indices and coordinates aside from the auxiliary field indices. Then we consider equations (77) for $\Delta E = \epsilon_k$ and the Hermitian conjugate equations for $\Delta E = \epsilon_j$. This gives

$$\epsilon_k \hat{\varphi}_n(k) = -i \sum_{l=1}^{n} \alpha^h(l) \partial_h(\boldsymbol{r}_l) \hat{\varphi}_n(k) \tag{79}$$

$$+ \sum_{l=1}^{n} \beta(l) \sum_{i_1 \ldots i_n} m_{i_l} \varphi_n(i_1 \ldots i_l \ldots i_n | k)$$

and

$$\epsilon_j \hat{\varphi}_n(j)^+ = i \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) \hat{\varphi}_n(j)^+ \alpha^h(l) \tag{80}$$

$$+ \sum_{l=1}^{n} \sum_{i_1 \ldots i_n} m_{i_l} \varphi_n(i_1 \ldots i_l \ldots i_n | j)^+ \beta(l)$$

Now we multiply (79) from the left hand side by $\hat{\varphi}_n(j)^+$ and (80) from the right hand side by $\hat{\varphi}_n(k)$, i.e. we form the (super) spinorial scalarproduct of these spintensors. Afterwards we subtract the resulting equation (80) from (79). This yields

$$(\epsilon_k - \epsilon_j)[\hat{\varphi}_n(j)^+ \hat{\varphi}_n(k)] = \tag{81}$$

$$-i \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\varphi}_n(j)^+ \alpha^h(l) \hat{\varphi}_n(k)]$$

$$+ \sum_{l=1}^{n} \hat{\varphi}_n(j)^+ \beta(l) \sum_{i_1 \ldots i_n} m_{i_l} \varphi_n(i_1 \ldots i_l \ldots i_n | k)$$

$$- \sum_{l=1}^{n} \sum_{i_1 \ldots i_n} m_{i_l} \varphi_n(i_1 \ldots i_l \ldots i_n | j)^+ \beta(l) \hat{\varphi}_n(k).$$

Then with $m_i = m + \delta m_i$ equation (81) goes over into

$$(\epsilon_k - \epsilon_j)[\hat{\varphi}_n(j)^+ \hat{\varphi}_n(k)] = \tag{82}$$

$$-i \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\varphi}_n(j)^+ \alpha^h(l) \hat{\varphi}_n(k)]$$

$$+ \sum_{l=1}^{n} \sum_{i_1 \ldots i_n} [\hat{\varphi}_n(j)^+ \beta(l) \varphi_n(i_1 \ldots i_l \ldots i_n | k) \delta m_{i_l}$$

$$- \varphi_n(i_1 \ldots i_l \ldots i_n | j)^+ \beta(l) \hat{\varphi}_n(k) \delta m_{i_l}]$$

Provided that $\sum_{\substack{i_1 \ldots i_n \\ \neq i_l}} \varphi_n(i_1 \ldots i_l \ldots i_n)$ contains no divergent terms and that the physical functions $\hat{\varphi}_n$ are regular, the last term in (82) vanishes in the limit of vanishing auxiliary field mass differences δm_i. But the essential point is that such a limit procedure is only allowed after all calculations were performed. In this case this limit means the transition to a massive subfermion and a massive subfermion dipole ghost at the same (high) mass value. Then one gets from (82)

$$(\epsilon_k - \epsilon_j) [\hat{\varphi}_n(j)^+ \hat{\varphi}_n(k)] = \tag{83}$$

$$-i \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\varphi}_n(j)^+ \alpha^h(l) \hat{\varphi}_n(k)],$$

and integration over $\boldsymbol{r}_1 \ldots \boldsymbol{r}_n$ leads to

$$(\epsilon_k - \epsilon_j) \int \hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \ldots \boldsymbol{r}_n \\ z_1 \ldots z_n \end{pmatrix} j \end{pmatrix}^+ \hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \ldots \boldsymbol{r}_n \\ z_1 \ldots z_n \end{pmatrix} k \end{pmatrix}$$

$$\cdot \, \mathrm{d}^3 r_1 \ldots \mathrm{d}^3 r_n = -i \sum_{l=1}^{n} \int \partial_h(\boldsymbol{r}_l) [\hat{\varphi}_n(j) \alpha^h(l) \hat{\varphi}_n(k)]$$

$$\cdot \, \mathrm{d}^3 r_1 \ldots \mathrm{d}^3 r_n = 0. \tag{84}$$

The right hand side of (84) is a $\mathrm{I\!R}^{3n}$ highdimensional divergence in the integrand and vanishes if the physical amplitudes $\hat{\varphi}_n(k)$ satisfy corresponding boundary conditions at infinity. This assumption corresponds to the physical situation of local processes which even for scattering processes in the laboratory has to be satisfied. Then one has to conclude that $\hat{\varphi}_n(j)$ and $\hat{\varphi}_n(k)$ are orthogonal with respect to the inner product of a Hilbert space.

6. Regularization and Probability Interpretation

In this section we continue the investigation of the state space of physical amplitudes. This inner product space cannot be the Krein space of the auxiliary field states, as by definition these states are unphysical states.

As for a closed system without external forces the inner product structure has to be selfconsistently defined in accordance with the system dynamics, the only way to proceed is the use of the dynamical equations. In order to do this we consider the time dependent eigenstates

$$
\hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n, t \\ z_1 \dots z_n \end{pmatrix} := e^{-i\epsilon_k t} \hat{\varphi}_n \begin{pmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n \\ z_1 \dots z_n \end{pmatrix} k \Big). \quad (85)
$$

By definition these states are derived from the auxiliary field states

$$
\varphi_n \begin{pmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n, t \\ z_1 \dots z_n \\ i_1 \dots i_n \end{pmatrix} := e^{-i\epsilon_k t} \varphi_n \begin{pmatrix} \boldsymbol{r}_1 \dots \boldsymbol{r}_n \\ z_1 \dots z_n \\ i_1 \dots i_n \end{pmatrix} k \Big),
$$
$$
(86)
$$

i. e. the spectrum of the physical states is identical with the spectrum of the auxiliary field states, a fact which follows directly from (77).

Then (79) can be transformed into an equation for time dependent eigenstates, which is given by

$$
\frac{\partial}{\partial t} \hat{\varphi}_n(t) = - \sum_{l=1}^{n} \alpha^h(l) \partial_h(\boldsymbol{r}_l) \hat{\varphi}_n(t), \quad (87)
$$

$$
- i \sum_{l=1}^{n} \beta(l) \sum_{i_1 \dots i_n} m_{i_l} \varphi_n(i_1 \dots i_l \dots i_n | t),
$$

and the Hermitian conjugate equation reads

$$
\frac{\partial}{\partial t} \hat{\varphi}_n(t)^+ = - \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) \hat{\varphi}_n(t)^+ \alpha^h(l) \quad (88)
$$

$$
+ i \sum_{l=1}^{n} \sum_{i_1 \dots i_n} m_{i_l} \varphi_n(i_1 \dots i_l \dots i_n | t)^+ \beta(l).
$$

Now we multiply (87) from the left hand side by $\hat{\varphi}_n(t)^+$ and (88) from the right hand side by $\hat{\varphi}_n(t)$. Addition of both expressions gives

$$
\frac{\partial}{\partial t} [\hat{\varphi}_n(t)^+ \hat{\varphi}_n(t)] + \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\varphi}_n(t)^+ \alpha^h(l) \hat{\varphi}_n(t)]
$$

$$
+ i \sum_{l=1}^{n} \hat{\varphi}_n(t)^+ \beta(l) \sum_{i_1 \dots i_n} m_{i_l} \varphi_n(i_1 \dots i_l \dots i_n | t) \quad (89)
$$

$$
- i \sum_{l=1}^{n} \sum_{i_1 \dots i_n} m_{i_l} \varphi_n(i_1 \dots i_l \dots i_n | t)^+ \beta(l) \hat{\varphi}_n(t) = 0.
$$

This result can be generalized to the case where instead of single eigenstates (85) linear combinations of them are used. By such combinations (87) is fulfilled too. Therefore in the following we consider time dependent states given by

$$
\phi_n(i_1 \dots i_n, t) = \sum_k c_k e^{-i\epsilon_k t} \varphi_n(i_1 \dots i_n | k) \quad (90)
$$

We substitute the corresponding functions into (89) and obtain

$$
\frac{\partial}{\partial t} [\hat{\phi}_n(t)^+ \hat{\phi}_n(t)] + \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\phi}_n(t)^+ \alpha^h(l) \hat{\phi}_n(t)]
$$

$$
+ i \sum_{l=1}^{n} \sum_k c_k^* e^{i\epsilon_k t} \hat{\varphi}_n(k)^+ \beta(l) \sum_{k'} \sum_{i_1 \dots i_n} m_{i_l} \quad (91)
$$

$$
\cdot \varphi_n(i_1 \dots i_l \dots i_n | k') c_{k'} e^{-i\epsilon_{k'} t}
$$

$$
- i \sum_{l=1}^{n} \sum_k \sum_{i_1 \dots i_n} m_{i_l} \varphi_n(i_1 \dots i_l \dots i_n | k)^+ c_k^* e^{i\epsilon_k t} \beta(l)
$$

$$
\cdot \sum_{k'} c_{k'} e^{-i\epsilon_{k'} t} \hat{\varphi}_n(k') = 0
$$

Under the same assumptions as in Sect 5 we can perform the limit $\delta m_i \to 0$ and get the final equation

$$
\frac{\partial}{\partial t} [\hat{\phi}_n(t)^+ \hat{\phi}_n(t)] + \sum_{l=1}^{n} \partial_h(\boldsymbol{r}_l) [\hat{\phi}_n(t)^+ \alpha^h(l) \hat{\phi}_n(t)] = 0,
$$
$$
(92)
$$

i. e. continuity equations for $n = 1 \dots \infty$. From these equations it follows that $\hat{\phi}_n(t)^+ \hat{\phi}_n(t)$ has to be interpreted as a positive definite probability density which leads to probability conservation if the amplitudes $\hat{\phi}_n(t)$ satisfy corresponding boundary conditions at infinity.

Nevertheless this result is somewhat disturbing as instead of one continuity equation for the whole state one gets separate continuity equations for each component. To show that this leads to no contradiction we observe that the set of stationary amplitudes $\{\hat{\varphi}_n(\boldsymbol{r}_1 \dots \boldsymbol{r}_n | k), \; n = 1 \dots \infty\}$ belongs to one state. This means that if the set of amplitudes $\{\varphi_n(I_1 \dots I_n | k), \; n = 1 \dots \infty\}$ is a solution of (73), then also $\{\mathcal{N}_k \varphi_n(I_1 \dots I_n | k), \; n = 1 \dots \infty\}$ is a solution of (73), i. e. the original set can only be multiplied

by one common constant $\mathcal{N}_k$ without destroying its property to be a solution of (73).

Therefore by definition let us consider the normalization integrals of the original set $\{\hat{\varphi}_n(k)\}$

$$\int \hat{\varphi}_n \left(\begin{array}{c} r_1 \ldots r_n \\ z_1 \ldots z_n \end{array} \bigg| k \right)^* \hat{\varphi}_n \left(\begin{array}{c} r_1 \ldots r_n \\ z_1 \ldots z_n \end{array} \bigg| k \right) \mathrm{d}^{3n}\tau = a_k^n. \tag{93}$$

Then the set $\{\mathcal{N}_k \hat{\varphi}_n(k)\}$ leads to the normalization

$$|\mathcal{N}_k|^2 \int \hat{\varphi}_n \left(\begin{array}{c} r_1 \ldots r_n \\ z_1 \ldots z_n \end{array} \bigg| k \right)^* \hat{\varphi}_n \left(\begin{array}{c} r_1 \ldots r_n \\ z_1 \ldots z_n \end{array} \bigg| k \right) \mathrm{d}^{3n}\tau$$

$$= |\mathcal{N}_k|^2 a_k^n. \tag{94}$$

From this it follows that the only way to normalize the state $|k\rangle$ is given by the condition

$$\sum_{n=1}^{\infty} |\mathcal{N}_k|^2 a_k^n = 1. \tag{95}$$

We apply this condition to (92). We integrate (92) over $\mathbb{R}^{3n}$. In accordance with the assumed boundary conditions we then get with (93) and (84)

$$\frac{\partial}{\partial t} \int \left[\hat{\phi}_n(t)^+ \hat{\phi}_n(t) \right] \mathrm{d}^{3n}\tau \tag{96}$$

$$= \frac{\partial}{\partial t} \sum_{kk'} \int c_k^* e^{i\epsilon_k t} c_{k'} e^{-i\epsilon_{k'} t} \hat{\varphi}_n(r_1 \ldots r_n | k)^+$$

$$\cdot \hat{\varphi}_n(r_1 \ldots r_n | k') \mathrm{d}^{3n}\tau = \frac{\partial}{\partial t} \sum_k |c_k|^2 a_k^n = 0$$

If in the expansion (90) normalized amplitudes are used, then (96) leads to

$$\frac{\partial}{\partial t} \sum_k |c_k|^2 |\mathcal{N}_k|^2 a_k^n = 0, \tag{97}$$

and in accordance with (95) summation over n yields

$$\frac{\partial}{\partial t} \sum_k |c_k|^2 = 0, \tag{98}$$

i. e. probability conservation for arbitrary time dependent amplitudes (90) referred to normalized states.

7. Physical State Space Construction

In the preceding sections we demonstrated that by means of the auxiliary field dynamics regularized

physical field amplitudes can be derived for which the interpretation and conservation of probability can be proved. However the preceding calculations leave the question undecided whether the formally defined physical amplitudes really do exist in particular with respect to the fact that for these regularized amplitudes no algebraic Schrödinger representation can be selfconsistently calculated. In the following we discuss how by means of covariant equations for time ordered matrixelements this problem can be solved.

Whithout trying to give a full treatment of this subject, we concentrate on the development of composite particle theory i. e. we study representation spaces which correspond to the description of composite particle processes. In this case some definitive statements about the existence of the representation can be made. As the corresponding results were already published, see for instance [9] etc. we refer to the literature and give only a summary which emphasizes the essential points of view.

We start with the definition of single composite particle states. We define hard core bound states for fermion numbers $n = 2, 3, \ldots$ and quantum numbers p as solutions of the diagonal part of (66):

$$E_{p0} |\mathcal{F}(j^t; p)\rangle^{\mathrm{d}} = \left(j_{I_1}^t \hat{K}_{I_1 I_2} \partial_{I_2}^t - 3\hat{W}_{I_1 I_2 I_3 I_4} \right. \tag{99}$$

$$\left. \cdot F_{I_4 K}^t j_{I_1}^t j_K^t \partial_{I_3}^t \partial_{I_2}^t \right) |\mathcal{F}(j^t; p)\rangle^{\mathrm{d}},$$

where the single composite particle quantum numbers p have to be sharply distinguished from the quantum numbers a of the full states (64). Due to the algebraic degrees of freedom which are contained in the symbolic indices I, (99) is rather complicated even for the low fermion numbers $n = 2, 3, 4$. However, these complications can be reduced if (99) is related to its covariant counterpart equation. This is a special feature of quantum field theory and it may be used to obtain solutions of (99) via the solution of the associated covariant equation with covariant sources j_k, see [9]

$$K_{I_1 I_2} \partial_{I_2} |\mathcal{F}(j; p)\rangle^{\mathrm{d}} = 3W_{I_1 I_2 I_3 I_4} \tag{100}$$

$$\cdot F_{I_4 K} j_K \partial_{I_3} \partial_{I_2} |\mathcal{F}(j; p)\rangle^{\mathrm{d}}$$

where according to (43) - (46) and (70)

$$K_{I_1 I_2} = \left[D_{Z_1 Z_2}^\mu \partial_\mu(x_1) - m_{Z_1 Z_2} \right] \delta(x_1 - x_2), \tag{101}$$

$$W_{I_1 I_2 I_3 I_4} = \hat{U}_{Z_1 [Z_2 Z_3 Z_4]} \delta(x_1 - x_2)\delta(x_1 - x_3)\delta(x_1 - x_4) \tag{102}$$

$$F_{I_1 I_2} = i\lambda_{i_1} \delta_{i_1 i_2} \gamma^5_{\kappa_1 \kappa_2} \left[(i\gamma^\mu \partial_\mu(x_1) + m_{i_1}) C \right]_{\alpha_1 \alpha_2} \quad (103)$$

$$\cdot \Delta \left(x_1 - x_2, m_{i_1} \right)$$

with the superindex κ defined by $\kappa = (A, \Lambda)$, and where $\Delta \left(x_1 - x_2, m_{i_1} \right)$ is the Feynman propagator

$$\Delta (x, m) := (2\pi)^{-4} \int \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon} \, \mathrm{d}^4 p.$$

Equation (100) admits separate solutions in any fermion number sector. Thus for the bound state functionals we apply the ansatz

$$| \mathcal{F}(j; p, n) \rangle^{\mathrm{d}} := \frac{i^n}{n!} \varphi^{(n)} \left(I_1 \dots I_n | p \right) j_{I_1} \dots j_{I_n} |0\rangle_F.$$
$$(104)$$

For simplicity we confine ourselves to the description of boson processes with composite bosons from n=2 states in (104), which were already thoroughly studied cf. Stumpf and Borne [9]. We first discuss the results for the single boson states.

i) Bound states of two subfermions (subfermion fields) can be exactly calculated in the auxiliary field formalism. These bound states have to transform themselves in a relativistically covariant way. Therefore their defining equations must be covariant, too. They are given by (100) with the associated energy equation (99). The relation between covariant solutions of (100) and noncovariant (one-time) solutions of (99) is established by Prop. 6.2 in [9].

The exact solutions of (100) for vector boson states are derived by Prop. 6.4 in [9]. From this proposition it follows that the associated secular equation for the vector boson mass μ_B can be exclusively formulated by means of fully regularized Green functions and propagators (71), in spite of the use of the auxiliary field formalism. For the explicit numerical evaluation of this secular equation, which leads to finite values of μ_B, we refer to the thesis by Sand [23]. The corresponding exact vector boson states depend on the auxiliary field indices, and as a consequence of this they contain singular distributions. In contrast to that, the physical amplitudes $\hat{\varphi}^{(2)}$ are fully regularized, as can be verified by summation over i_1, i_2, and the eigenvalues do not depend on the auxiliary field indices, provided we consider bound state solutions.

Now we turn to the full theory. To describe boson processes the original spinorfield theory has to be transformed into a theory for composite boson states. The description of bosons as bound states of subfermions and the derivation of a corresponding effective dynamics has to start from the normalordered energy equation on the hyperplane (66). This was justified in detail in [9]. Hence the intended map has to be applied to (66). An inspection of this equation shows that the set of solutions can be subdivided into two classes which completely decouple: those with all coeffecient functions of $|\mathcal{F}(j^t; a)\rangle$ belonging to an even number of arguments, and those with an odd number of arguments. With respect to the mapping onto a boson theory for two-particle bound boson states only the first class of solutions is of interest. Thus we confine our considerations to the functional states

$$|\mathcal{F}(j; a)\rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \varphi^{(2n)}(I_1 \dots I_{2n} | a) \quad (105)$$

$$\cdot j_{I_1} \dots j_{I_{2n}} |0\rangle_F$$

on the hyperplane $t = \text{const}$.

ii) In the auxiliary field formalism the one-time normalordered state amplitudes $\varphi^{(n)}$ are assumed to be representable by expansions into series of m-particle field states, $m < n$. Such expansions can (to some extent) be justified by inserting intermediate states. For instance the corresponding series expansions for $m = 2$ are given by

$$\varphi^{(2n)}(I_1 \dots I_{2n} | a) = \sum_{k_1 \dots k_n} C_{k_1}^{[I_1 I_2} \dots C_{k_n}^{I_{2n-1} I_{2n}]} \quad (106)$$

$$\cdot \rho^{(n)}(k_1 \dots k_n | a),$$

where $\left\{ C_k^{II'} \right\}$ is a complete set of two-particle states derived from the solutions of the hard core equation and $[I_1 \dots I_{2n}]$ means antisymmetrization.

Hence the singularities of $\varphi^{(2n)}$ are expressed by the singularities of their substates, i. e. of the $C_k^{II'}$. The occurrence of such singularities in the set $\left\{ \varphi^{(2n)} \right\}$, however, does not disturb the derivation of an effective dynamics for these composite states. In order to guarantee a finite map, dual states (test functions) are introduced (see [9]) which completely compensate the singularities of the original states and lead to finite expressions of the effective dynamics. Equation (106) establishes a map $\rho^{(n)} \rightarrow \varphi^{(2n)}$. If the set $\left\{ C_k^{II'} \right\}$ is complete the inverse mapping $\varphi^{(2n)} \rightarrow \rho^{(n)}$ exists,

i. e., the map is bijective, provided only symmetrized $\rho^{(n)}(k_1 \ldots k_n)$ are admitted. The connection to effective field equations is established by interpreting the boson coefficient functions $\rho^{(n)}$ as symmetrized expectation values of effective boson field operators; for a detailed discussion we refer to [9].

Equation (106) provides additional evidence for the use of (66) as the starting point of Weak Mapping. One can expect the set of functions $C_k^{I_1 I_2}$ to be complete only on a hyperplane $t = $ const. if one rejects Euclidean field theories as unphysical and compares the functions $C_k^{I_1 I_2}$ with ordinary quantum mechanical wave functions. Only if due to completeness relations the map (106) is bijective, (106) allows a faithful mapping of the basic field dynamics and prevents an uncontrollable loss of information of the basic theory. Thus the need for completeness enforces the use of the noncovariant formalism for the meaningfull definition of Weak Mapping.

In spite of the independence of Weak Mapping from the functional space, we want to formulate the *results* of Weak Mapping by functional equations. This was done in the following proposition derived by Pfister and Stumpf [24]. This proposition is an alternative version to a theorem of Kerschner and Stumpf [25].

Proposition 7.1: By the transformation (106) the functional energy equation (66) of the fermionic spinorfield (34) is mapped onto the following functional boson energy equation:

$$E_{p0}|\tilde{\mathcal{B}}(b;a)\rangle = 2R_{I_1 K}^k \hat{K}_{I_1 I_2} C_{k'}^{I_2 K} b_k \partial_{k'} |\tilde{\mathcal{B}}(b;a)\rangle \quad (107)$$

$$- \hat{W}_{K_1 I_2 I_3 I_4} \Big\{ 6F_{I_4 K_2} R_{K_1 K_2}^k C_{k'}^{I_2 I_3} b_k \partial_{k'}$$

$$- 36 F_{I_4 K_2} R_{K_1 K_3}^{k_1} R_{K_2 K_4}^{k_2} C_{k_1'}^{[I_2 I_3} C_{k_2'}^{K_3 K_4]} b_{k_1} b_{k_2} \partial_{k_1'} \partial_{k_2'}$$

$$- 6 R_{K_1 K_2}^k C_{k_1'}^{[I_2 I_3} C_{k_2'}^{I_4 K_2]} b_k \partial_{k_1'} \partial_{k_2'} - \big(3 F_{I_4 K_2} F_{I_3 K_3}$$

$$+ \frac{1}{4} A_{I_4 K_2} A_{I_3 K_3}\big) \sum_{p \in S_3} \text{sgn}(p) \frac{1}{3!} \big[12 R_{K_{p_1} K_{p_2}}^{k_1} R_{K_{p_3} K_4}^{k_2}$$

$$\cdot C_{k_1'}^{I_2 K_4} b_{k_1} b_{k_2} \partial_{k'} + 24 R_{K_{p_1} K_4}^{k_1} R_{K_{p_2} K_5}^{k_2} R_{K_{p_3} K_6}^{k_3} C_{k_1'}^{[I_2 K_6}$$

$$\cdot C_{k_2'}^{K_5 K_4]} b_{k_1} b_{k_2} b_{k_3} \partial_{k_1'} \partial_{k_2'} \big] + \big(F_{I_4 K_2} F_{I_3 K_3} + \frac{1}{4} A_{I_4 K_2}$$

$$\cdot A_{I_3 K_3}\big) F_{I_2 K_4} \sum_{p \in S_4} \text{sgn}(p) \frac{1}{4!} \big[12 R_{K_{p_1} K_{p_2}}^{k_1} R_{K_{p_3} K_{p_4}}^{k_2} b_{k_1} b_{k_2}$$

$$+ 48 R_{K_{p_1} K_5}^{k_1} R_{K_{p_2} K_6}^{k_2} R_{K_{p_3} K_{p_4}}^{k_3} C_{k'}^{K_5 K_6} b_{k_1} b_{k_2} b_{k_3} \partial_{k'}$$

$$+ 48 R_{K_{p_1} K_5}^{k_1} \ldots R_{K_{p_4} K_8}^{k_4} C_{k_1'}^{[K_5 K_6} C_{k_2'}^{K_7 K_8]}$$

$$\cdot b_{k_1} \ldots b_{k_4} \partial_{k_1'} \partial_{k_2'} \big] \Big\} |\tilde{\mathcal{B}}(b;a)\rangle$$

with

$$|\tilde{\mathcal{B}}(b;a)\rangle := \mathcal{U}|\mathcal{B}(b;a)\rangle, \quad (108)$$

$$\mathcal{B}(b;a)\rangle := \sum_{n=0}^{\infty} \frac{1}{n!} \rho^{(n)}(k_1 \ldots k_n \mid a) b_{k_1} \ldots b_{k_n} |0\rangle_\mathrm{B}, \quad (109)$$

and

$$\mathcal{U} := \sum_{n=0}^{\infty} \frac{1}{n!} b_{k_1'} \ldots b_{k_n'} |0\rangle_B R_{K_1 K_2}^{k_1'} \ldots R_{K_{2n-1} K_{2n}}^{k_n'}$$

$$\cdot C_{k_1}^{[K_1 K_2} \ldots C_{k_n}^{K_{2n-1} K_{2n}]}{}_\mathrm{B}\langle 0|\partial_{k_n} \ldots \partial_{k_1}, (110)$$

where the b_k and their duals $\partial_{k'}$ are Fockspace operators of a boson functional space which by definition is completely independent of the fermionic functional space used for generating functional states in (105).   □

For the rather extensive proof of Proposition 7.1 we refer to [9]. Without proof we remark that for comparison with the results by [24] and [9] the following identities have to be used:

$$\mathcal{V}\begin{pmatrix} I_1 I_2 \\ k \end{pmatrix} \partial_k |\mathcal{B}(b;a)\rangle = \sum_l C_l^{I_1 I_2} \partial_l |\tilde{\mathcal{B}}(b;a)\rangle \quad (111)$$

and

$$\mathcal{W}\begin{pmatrix} I_1 I_2 & I_3 I_4 \\ k & k' \end{pmatrix} \partial_k \partial_{k'} |\mathcal{B}(b;a)\rangle \quad (112)$$

$$= \sum_{ll'} C_l^{I_1 I_2} C_{l'}^{I_3 I_4} \partial_l \partial_{l'} |\tilde{\mathcal{B}}(b;a)\rangle,$$

where $\mathcal{V}$ and $\mathcal{W}$ are defined in [24] and [9].

Equation (107) represents a faithful mapping of the spinorfield dynamics onto a boson field dynamics and it contains all subtleties of the interactions of the fermionic constituents of the boson states under consideration. In order to extract the main physical effects from this equation we have to simplify and to approximate it in a suitable way. But this is not the topic of this paper.

We can summarize the results of nonperturbative Pauli-Villars regularization of the spinorfield model in the following way:

The fact that a quantum field theory allows various inequivalent representations is used to define a composite particle representation (105) (106) of the spinorfield state space. Due to a one-to-one mapping, Proposition 7.1 states: If the functional state $\mathcal{B}(b; a)\rangle$ (109) is a solution of (107), then the functional state $|\mathcal{F}(j; a)\rangle$ (105) must be a solution of (68) (69) if the relation (106) is postulated. In the limit of a monopole-dipole regularization, current conservation (92) holds for the associated regularized states (72), which arise from the states (105). The corresponding densities are for all a, i. e. for all defining quantum numbers integrable functions because due to (106) they contain products of regularized hard core functions which carry the whole coordinate dependence of the states under consideration and whose densities are integrable.

Furthermore, according to (106) the regularized $\hat{\varphi}^{(2n)}$ contain also the coefficient functions $\rho^{(n)}(k_1 \ldots k_n \mid a)$. These functions follow from (107). This equation results from the auxiliary field dynamics and is completely free of singular expressions because by introduction of dual states all singularities contained in the mapping (106) are compensated. Hence the $\rho^{(n)}$ should be finite too. Therefore the single composite particle bound states $\hat{\varphi}^{(m)}$ are elements of a corresponding Hilbert space in accordance with the field dynamics, and every $\hat{\varphi}^{(n)}$ which is composed of these substates is an element of a Hilbert space, too.

According to this summary we can assume that the norm expressions (93) and (94) really exist. But if this is the case, then (96) has a definite meaning and a probability interpretation of the physical state space is possible.

In the other attempts undertaken so far, emphasis was laid on a physical probability interpretation of the original ghost field theories themselves, see for instance the older book by Nagy [26] or the recent treatment of Faddeev-Popov ghosts etc. by Nakanishi and Ojima [27]. In our approach the auxiliary (ghost) field theory is considered as unphysical, and in accordance with our discussion the physical theory is separated from the unphysical theory by a noninvertible map.

On the other hand, the unphysical theory is indispensible as it provides the basis for that map, and the new feature of this approach is the close connection of regularization, probability conservation and composite particle dynamics.

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